

RECONSTRUCTION THRESHOLD FOR THE HARDCORE MODEL

NAYANTARA BHATNAGAR, ALLAN SLY, AND PRASAD TETALI

ABSTRACT. In this paper we consider the reconstruction problem on the tree for the hardcore model. We determine new bounds for the non-reconstruction regime on the k -regular tree showing non-reconstruction when

$$\lambda < \frac{(\ln 2 - o(1)) \ln^2 k}{2 \ln \ln k}$$

improving the previous best bound of $\lambda < e - 1$. This is almost tight as reconstruction is known to hold when $\lambda > (e + o(1)) \ln^2 k$. We discuss the relationship for finding large independent sets in sparse random graphs and to the mixing time of Markov chains for sampling independent sets on trees.

1. INTRODUCTION

The reconstruction problem on the tree was originally studied as a problem in statistical physics but has since found many applications including in computational phylogenetic reconstruction [8], the study of the geometry of the space of random constraint satisfaction problems [1, 14] and the mixing time of Markov chains [5, 17]. For a Markov model on an infinite tree the reconstruction problem asks when do the states at level n provide non-trivial information about the state at the root as n goes to infinity. In general the problem involves determining the existence of solutions of distribution valued equations and as such exact thresholds are known only in a small number of examples [4, 10, 5, 25].

In this paper we analyze the reconstruction problem for the hardcore model on the k -regular tree, where each vertex of the tree has degree k . The hardcore model is a probability distribution over independent sets I weighted proportionally to $\lambda^{|I|}$. Previously Brightwell and Winkler [7] showed that reconstruction is possible when $\lambda > (e + o(1)) \ln^2 k$. Improving on their bound for the non-reconstruction regime, Martin [16] showed that non-reconstruction holds when $\lambda < e - 1$ still leaving a wide gap between the two thresholds. Our main result establishes that the bound of Brightwell and Winkler is tight up to a $\ln \ln k$ multiplicative factor.

Theorem 1.1. *The hardcore model on the k -regular tree has non-reconstruction when*

$$\lambda < \frac{(\ln 2 - o(1)) \ln^2 k}{2 \ln \ln k}.$$

Research supported in part by NSF grants DMS-0701043 and-CCR 0910584.

1.1. The Hardcore Model. For a finite graph G the independent sets $I(G)$ are subsets of the vertices containing no adjacent vertices. The hardcore model is a probability measure over $\sigma \in I(G) \subset \{0, 1\}^G$ such that

$$(1) \quad \mathbb{P}(\sigma) = \frac{1}{Z} \lambda^{\sum_{v \in G} \sigma_v} \mathbb{1}_{\sigma \in I(G)}$$

where λ is the *fugacity* parameter and Z is a normalizing constant. The definition of the hardcore model can be extended to infinite graphs by way of the Dobrushin-Lanford-Ruelle condition which essentially says that for every finite set A the configuration on A is given by the Gibbs distribution given by a random boundary generated by the measure outside of A . Such a measure is called a Gibbs measure and there may be one or infinitely many such measures (see e.g. [12] for more details). For every λ , there exists a unique translation invariant Gibbs measure on the k -regular tree and it is this measure which we study.

An alternative equivalent formulation of the hardcore model is as a Markov model on the k -regular tree. An independent set σ is generated by first choosing the root according to the distribution

$$(\pi_1, \pi_0) = \left(\frac{\omega}{1+2\omega}, \frac{1+\omega}{1+2\omega} \right)$$

for some $0 < \omega < 1$. The states of the remaining vertices of the graph are generated from their parents' states by taking one step of the Markov transition matrix

$$M = \begin{pmatrix} p_{11} & p_{10} \\ p_{01} & p_{00} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\omega}{1+\omega} & \frac{1}{1+\omega} \end{pmatrix}.$$

It can easily be checked that π is reversible with respect to M and that this generates a translation invariant Gibbs measure on the tree with fugacity

$$\lambda = \omega(1+\omega)^{k-1}.$$

Restating Theorem 1.1 in terms of ω we have non-reconstruction when

$$(2) \quad \omega \leq \frac{1}{k} \left[\ln k + \ln \ln k - \ln \ln \ln k - \ln 2 + \ln \ln 2 - o(1) \right] =: \bar{\omega}.$$

In contrast, from [7], for every fixed $\varepsilon > 0$ and k large enough, reconstruction is known to hold when

$$(3) \quad \omega \geq \frac{1}{k} \left[\ln k + \ln \ln k + 1 + \varepsilon \right].$$

We will introduce some further notation which we will make use of in the proof.

$$\begin{aligned} \pi_{01} &\equiv \frac{\pi_0}{\pi_1} = \frac{1+\omega}{\omega}, & \Delta &\equiv \pi_{01} - 1 = \frac{1}{\omega}, \\ \theta &\equiv p_{00} - p_{10} = p_{11} - p_{01} = -\frac{\omega}{1+\omega}. \end{aligned}$$

A particularly important role is played by θ , the second eigenvalue of M as is discussed in the following subsection. We denote by $\mathbb{P}_T^1, \mathbb{E}_T^1$ (and resp. $\mathbb{P}_T^0, \mathbb{E}_T^0$ and

$\mathbb{P}_T, \mathbb{E}_T$) the probability and expectations with respect to the measure obtained by conditioning on the root ρ of T to be 1 (resp. 0, and stationary). We let $L = L(n)$ denote the set of vertices at depth n and $\sigma(L) = \sigma(L(n))$ denote the configuration on level n . We will write $\Pr_T[\cdot | \sigma(L) = A]$ to denote the measure conditioned on the leaves being in state $A \in \{0, 1\}^{L(n)}$.

1.2. The reconstruction problem. The reconstruction problem on the tree essentially asks if we can recover information on the root from the spins deep inside the tree. In particular we say that the model has *non-reconstruction* if

$$(4) \quad \mathbb{P}_T[\sigma_\rho = 1 | \sigma(L)] \rightarrow \pi_1$$

in probability as $n \rightarrow \infty$, otherwise the model has *reconstruction*. Note that if we do not condition on the configuration at the leaves, the probability above is exactly π_1 by the Markov model formulation of the hardcore measure. Equivalent formulations of non-reconstruction are that the Gibbs measure is extremal or that the tail σ -algebra of the Gibbs measure is trivial [23]. It follows from Proposition 12 of [22] that there exists a λ_R such that reconstruction holds for $\lambda > \lambda_R$ and non-reconstruction holds for $\lambda < \lambda_R$. The reconstruction problem is to determine the threshold λ_R .

1.3. Related Work. A significant body of work has been devoted to the reconstruction problem on the tree by probabilists, computer scientists and physicists. The earliest such result is the Kesten-Stigum bound [15] which states that reconstruction holds whenever $\theta^2(k-1) > 1$. This bound was shown to be tight in the case of the Ising model [4, 10] where it was shown that non-reconstruction holds when $\theta^2(k-1) \leq 1$. Similar results were derived for the Ising model with small external field [2] and the 3-state Potts model [25] which constitute the only models for which exact thresholds are known. On the other hand, at least when k is large, the Kesten-Stigum bound is known not to be tight for the hardcore model [7]. As such, the most one can reasonably ask to show is the asymptotics of the reconstruction threshold $\lambda_R(k)$ for large k .

The Kesten-Stigum bound is known to be the correct bound for robust reconstruction for all Markov models [13]. Robust reconstruction asks whether reconstruction is possible after adding a large amount of noise to the spins in level n . It was shown in [13] that when $\theta^2(k-1) < 1$ after adding enough noise to the spins at level n , the “information” provided by the modified spins at level n decays exponentially quickly.

In both the coloring model and the hardcore model the reconstruction threshold is far from the Kesten-Stigum bound for large k . In the case of the hardcore model $\theta^2(k-1) = (1+o(1))\frac{1}{k} \ln^2 k$. As such, given a noisy version of the spins at level n , the information on the root decays rapidly as n grows. In the coloring model close to optimal bounds [3, 24] were obtained by first showing that, when n is small, the information on the root is sufficiently small. Then a quantitative version of [13] establishes that the information on the root converges to 0 exponentially

quickly. The hardcore model behaves similarly. Indeed, the form of our bound in equation (2) is strikingly similar to the bound for the q -coloring model which states that reconstruction (resp. non-reconstruction) holds when the degree is at least (resp. at most) $q[\ln q + \ln \ln q + O(1)]$.

Our proof then proceeds as follows. We first establish that when ω satisfies (2) then even for a tree of depth 3 there is already significant loss of information of the spin at the root. In particular we show that if the state of the root is 1 then the typical posterior probability that the state of the root is 1 given the spins at level 3 will be less than $\frac{1}{2}$. The result is completed by linearizing the standard tree recursion as in [5, 25]. In this part of the proof we closely follow the notation of [5] who analyzed the reconstruction problem for the Ising model with small external field. We do not require the full strength of their analysis as in our case we are far from the Kesten-Stigum bound. We show that a quantity which we refer to as the *magnetization* decays exponentially fast to 0. The magnetization provides a bound on the posterior probabilities and this completes the result.

The $\ln \ln k$ term in our bound on λ is explained as the first point at which there is significant decay of information at level 3 on the tree. In particular the analysis in Proposition 2.3 part c) is essentially tight. It may be possible to get improved bounds by considering higher depth trees although the description of the posterior distribution necessarily becomes more complex. A sharper analysis of this sort was done in [24] for the coloring model although the method there made crucial use of the symmetry of the states.

Replica Symmetry Breaking and Finding Large Independent Sets. The reconstruction problem plays a deep role in the geometry of the space of solutions of random constraint satisfaction problems. While for problems with few constraints the space of solutions is connected and finding solutions is generally easy, as the number of constraints increases the space may break into exponentially many small clusters. Physicists, using powerful but non-rigorous “replica symmetry breaking” heuristics, predicted that the clustering phase transition exactly coincides with the reconstruction region on the associated tree model [19, 14]. This picture was rigorously established (up to first order terms) for the coloring and satisfiability problems [1] and further extended to sparse random graphs by [20]. As solutions are far apart, local search algorithms will in general fail. Indeed for both the coloring and SAT models, no algorithm is known to find solutions in the clustered phase. It has been conjectured to be computationally intractable beyond this phase transition [1].

The associated CSP for the hardcore model corresponds to finding large independent sets in random k -regular graphs. The replica heuristics again predict that the space of large independent sets should be clustered in the reconstruction regime. Specifically this refers to independent sets of size sn where $s > \pi_1(R)$, the density of 1’s in the hardcore model at the reconstruction threshold. It is known that the largest independent set is with high probability $\frac{(2-o(1)) \ln k}{k}n$ [6].

On the other hand the best known algorithm finds independent sets only of size $\frac{(1+o(1)) \ln k}{k} n$ which is equal to $\pi_1(R)n$ [27]. This is consistent with the physics predictions and it would be of interest to determine if the space of independent sets indeed exhibits the same clustering phenomena as colorings and SAT at the reconstruction threshold. Determining the reconstruction threshold more precisely thus has implications for the problem of finding large independent sets in random graphs.

Glauber Dynamics on trees. The reconstruction threshold plays a key role in the study of the rate of convergence of the Glauber dynamics Markov chain for sampling spin systems on trees. This problem has received considerable attention (see e.g. [2, 9, 17, 18, 26]) and in the case of the Ising model, the mixing time is known to undergo a phase transition from $\Theta(n \ln n)$ in the non-reconstruction regime to $n^{1+\Theta(1)}$ in the reconstruction regime [2]. In fact, the mixing time is $n^{1+\Theta(1)}$ for any spin system above the reconstruction threshold. A similar transition was shown to take place for the coloring model [26]. Sharp bounds of this type are not known for the hardcore model, however, it is predicted that the Glauber dynamics should again be $O(n \log n)$ in the non-reconstruction regime.

Phylogenetic Reconstruction. Phylogenetic reconstruction is an important problem in evolutionary biology [11]. The results of [8, 21] imply that for binary symmetric channels the amount of data needed for phylogenetic reconstruction is closely related to the corresponding reconstruction threshold on the tree. This sampling efficiency undergoes a phase transition from $O(\log(N))$ in the reconstruction regime to $N^{\Omega(1)}$ in the non-reconstruction regime, where N is the number of leaves (or species) at the bottom of the phylogenetic tree. The results of [5] for non-reconstruction also imply lower bounds on the sample complexity for phylogenetic reconstruction for asymmetric channels.

2. PROOF OF THEOREM 1.1

For ease of notation we establish our bounds for the k -ary tree (where each vertex has k children) instead of on the k -regular tree. It is not difficult to modify the recursion we will obtain for the k -ary tree to a recursion for the $(k+1)$ -regular tree, showing that non-reconstruction also holds in that case. Finally, we can show that non-reconstruction on the k -regular tree is equivalent to non-reconstruction on the $(k+1)$ -regular tree once we note that in equation (2) we have that $\bar{\omega}(k+1) - \bar{\omega}(k) = o(k)$ so the difference can be absorbed in the error term. Let \mathcal{T} denote the infinite k -ary tree and let T_n denote the restriction of \mathcal{T} to its first n levels.

Before reading further, it might help the reader to quickly recall the notation from the end of Section 1.1. As in [5] we analyze a random variable X which denotes *weighted magnetization of the root* which is a function of the leaf states

of the tree. We define $X = X(n)$ on T_n by

$$(5) \quad \begin{aligned} X &= \pi_0^{-1}[\pi_0 \mathbb{P}(\sigma_\rho = 1 | \sigma(L) = A) - \pi_1 \mathbb{P}(\sigma_\rho = 0 | \sigma(L) = A)] \\ &= \frac{1}{\pi_{01}} \left[\frac{\mathbb{P}[\sigma_\rho = 1 | \sigma(L) = A]}{\pi_1} - 1 \right]. \end{aligned}$$

Since $\mathbb{E}_T[\mathbb{P}[\sigma_\rho = 1 | \sigma(L) = A]] = \mathbb{P}[\sigma_\rho = 1] = \pi_1$, from the above expression, we have that $\mathbb{E}[X] = 0$. Also, $X \leq 1$ since $\mathbb{P}[\sigma_\rho = 1 | \sigma(L) = A] \leq 1$. We will make extensive use of the following second moments of the magnetization.

$$\overline{X} = \mathbb{E}_T[X^2], \quad \overline{X}_1 = \mathbb{E}_T^1[X^2], \quad \overline{X}_0 = \mathbb{E}_T^0[X^2].$$

The following equivalent definition of non-reconstruction is well known and follows from the definition in (4) using (5).

Proposition 2.1. *Non-reconstruction for the model (\mathcal{T}, M) is equivalent to*

$$\lim_{n \rightarrow \infty} \overline{X}(n) = 0,$$

where $\overline{X}(n) = \mathbb{E}_{T_n}[X^2]$.

In the remainder of the proof we derive bounds for \overline{X} . We begin by showing that already for a 3 level tree, \overline{X} becomes small. Then we establish a recurrence along the lines of [5] that shows that once \overline{X} is sufficiently small, it must converge to 0. As this part of the derivation follows the calculation in [5] we will adopt their notation in places. Non-reconstruction is then a consequence of Proposition 2.1. In the next lemma we determine some basic properties of X .

Lemma 2.2. *The following relations hold:*

- a) $\mathbb{E}_T[X] = \pi_1 \mathbb{E}_T^1[X] + \pi_0 \mathbb{E}_T^0[X] = 0$.
- b) $\overline{X} = \pi_1 \overline{X}_1 + \pi_0 \overline{X}_0$.
- c) $\mathbb{E}_T^1[X] = \pi_{01} \overline{X}$ and $\mathbb{E}_T^0[X] = -\overline{X}$.

Proof. Note that for any random variable which depends only on the states at the leaves, $f = f(A)$, we have $E_T[f] = \pi_1 \mathbb{E}_T^1[f] + \pi_0 \mathbb{E}_T^0[f]$. Parts a) and b) therefore follow since X is a random variable that is a function of the states at the leaves. For part c) we proceed as follows. The first and last equalities below follow from (5).

$$\begin{aligned} \mathbb{E}_T^1[X] &= \pi_{01}^{-1} \sum_A \mathbb{P}_T[\sigma(L) = A | \sigma_\rho = 1] \left(\frac{\mathbb{P}_T[\sigma_\rho = 1 | \sigma(L) = A]}{\pi_1} - 1 \right) \\ &= \pi_{01}^{-1} \sum_A \mathbb{P}_T[\sigma(L) = A] \frac{\mathbb{P}_T[\sigma_\rho = 1 | \sigma(L) = A]}{\pi_1} \left(\frac{\mathbb{P}_T[\sigma_\rho = 1 | \sigma(L) = A]}{\pi_1} - 1 \right) \\ &= \pi_{01}^{-1} \left(\frac{\mathbb{E}_T[(\mathbb{P}_T[\sigma_\rho = 1 | \sigma(L) = A])^2]}{\pi_1^2} - 1 \right) = \pi_{01} \mathbb{E}[X^2]. \end{aligned}$$

The second part of c) follows by combining this with a). □

The following proposition estimates typical posterior probabilities which we will use to bound \overline{X} . For a finite k -ary subtree T of the $k+1$ -regular tree let T^i be the subtrees rooted at the children of the root u_i .

Proposition 2.3. *For a finite k -ary subtree T we have that*

a) *For any configuration at the leaves $A = (A_1, \dots, A_k)$,*

$$\mathbb{P}_T[\sigma_\rho = 0 | \sigma(L) = A] = \left(1 + \lambda \prod_i \mathbb{P}_{T^i}[\sigma_{u_i} = 0 | \sigma_{L_i} = A_i]\right)^{-1}.$$

b) *Let \mathcal{G} be the set of leaf configurations*

$$\mathcal{G} = \left\{ \sigma(L) \mid \mathbb{P}[\sigma_\rho = 0 | \sigma(L)] = \frac{1}{2} \left(1 + \frac{1}{1 + 2\lambda}\right) \right\}.$$

Then

$$\frac{\mathbb{P}_T^0[\sigma(L) \in \mathcal{G}]}{\mathbb{P}_T^1[\sigma(L) \in \mathcal{G}]} = \frac{\pi_1}{\pi_0} \frac{1 + \lambda}{\lambda}.$$

c) *Let $\beta > \ln 2 - \ln \ln 2$ and $\omega = \frac{1}{k} [\ln k + \ln \ln k - \ln \ln \ln k - \beta]$. Then in the 3 level k -ary tree T_3 we have that*

$$\mathbb{E}_{T_3}^1[\mathbb{P}[\sigma_\rho = 1 | \sigma(L)]] \leq \frac{1}{2}.$$

Proof. Part a) is a consequence of standard tree recursions for Markov models established using Bayes rule.

For part b) first note that

$$\begin{aligned} \mathbb{P}[\sigma_\rho = 1 \mid \sigma(L) \in \mathcal{G}] &= 1 - \mathbb{P}[\sigma_\rho = 0 \mid \sigma(L) \in \mathcal{G}] \\ (6) \qquad \qquad \qquad &= \frac{1}{2} \left(1 - \frac{1}{1 + 2\lambda}\right). \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{P}_T^0[\sigma(L) \in \mathcal{G}] &= \frac{\mathbb{P}[\sigma_\rho = 0 \mid \sigma(L) \in \mathcal{G}] \mathbb{P}[\sigma(L) \in \mathcal{G}]}{\pi_0} \\ &= \frac{\pi_1}{\pi_0} \frac{1 + \lambda}{\lambda} \left(\frac{\mathbb{P}[\sigma_\rho = 1 \mid \sigma(L) \in \mathcal{G}] \mathbb{P}[\sigma(L) \in \mathcal{G}]}{\pi_1} \right) \\ &= \frac{\pi_1}{\pi_0} \frac{1 + \lambda}{\lambda} \mathbb{P}_T^1[\sigma(L) \in \mathcal{G}] \end{aligned}$$

where the first and third equations follow by definition of conditional probabilities and the second follows from (6) which establishes b).

For part c), we start by calculating the probability of certain posterior probabilities for trees of small depth. With our assumption on ω we have that

$$\lambda = \omega(1 + \omega)^k = \frac{(1 + o_k(1))e^{-\beta} \ln^2 k}{\ln \ln k}.$$

Since $\sigma(L) \equiv 0$ under $\mathbb{P}_{T_1}^1$, by part *a*) we have that

$$\mathbb{P}_{T_1}^1[\sigma_\rho = 0 | \sigma(L)] = \frac{1}{1 + \lambda} \text{ w.p. } 1.$$

Also,

$$\mathbb{P}_{T_1}(u_i = 0 \ \forall \ i | \sigma_\rho = 0) = \left(\frac{1}{1 + \omega} \right)^k.$$

Using the two equations above, we have that

$$\mathbb{P}_{T_1}^0(\sigma_\rho = 0 | \sigma(L)) = \begin{cases} 1 & \text{w.p. } 1 - \left(\frac{1}{1 + \omega} \right)^k \\ \frac{1}{1 + \lambda} & \text{w.p. } \left(\frac{1}{1 + \omega} \right)^k. \end{cases}$$

The first case above corresponds to leaf configurations of the tree T_1 where at least one of the leaves is 1, while the second case corresponds to the configurations where all the leaves are 0. Next, applying part *a*) to a tree of depth 2, we have

$$\mathbb{P}_{T_2}^1[\sigma_\rho = 0 | \sigma(L)] = \frac{1}{1 + \lambda \prod_i \mathbb{P}_{T_1}^0[\sigma_{u_i} = 0 | \sigma(L)]}.$$

Using this expression we can write down this conditional probability based on the leaf configurations of the subtrees of the root of depth 1.

$$(7) \quad \mathbb{P}_{T_2}^1[\sigma_\rho = 0 | \sigma(L)] = \begin{cases} \frac{1}{1 + \lambda} & \text{w.p. } \left(1 - \left(\frac{1}{1 + \omega} \right)^k \right)^k \\ \frac{1}{2} \left(1 + \frac{1}{1 + 2\lambda} \right) & \text{w.p. } \left(1 - \left(\frac{1}{1 + \omega} \right)^k \right)^{k-1} \left(\frac{1}{1 + \omega} \right)^k k \\ > \frac{1}{2} \left(1 + \frac{1}{1 + 2\lambda} \right) & \text{o.w.} \end{cases}$$

The first case above corresponds to the situation when each subtree of the root of depth 1 has a leaf configuration where at least one of the leaves is 1. The second case is when one of the k subtrees has a leaf configuration where all leaves are 0, while the remaining subtrees have leaf configurations where at least one leaf is 1. The third case corresponds to the remaining possibilities.

By part *b*) with \mathcal{G} as defined, and (7) we have that after substituting the expressions for λ and ω ,

$$(8) \quad \begin{aligned} \mathbb{P}_{T_2}^0[\sigma(L) \in \mathcal{G}] &= \frac{\pi_1}{\pi_0} \frac{1 + \lambda}{\lambda} \mathbb{P}_{T_2}^1[\sigma(L) \in \mathcal{G}] \\ &= \frac{\omega(1 + \lambda)}{\lambda(1 + \omega)} \left(1 - \left(\frac{1}{1 + \omega} \right)^k \right)^{k-1} \left(\frac{1}{1 + \omega} \right)^k k \\ &\geq (1 - o_k(1)) \frac{e^{\beta \ln \ln k}}{k}. \end{aligned}$$

We can now calculate the values of $\mathbb{P}_{T_3}^1[\sigma_\rho = 0 | \sigma(L)]$ as follows. By part *a*)

$$\mathbb{P}_{T_3}^1[\sigma_\rho = 0 | \sigma(L)] = \frac{1}{1 + \lambda \prod_i \mathbb{P}_{T_2}^0[\sigma_{u_i} = 0 | \sigma(L)]}.$$

Denote

$$p = \frac{\omega(1+\lambda)}{\lambda(1+\omega)} \left(1 - \left(\frac{1}{1+w} \right)^k \right)^{k-1} \left(\frac{1}{1+\omega} \right)^k k.$$

Thus, p is the probability that if we started with $\sigma_\rho = 0$ in T_2 , the configuration at the leaves is from \mathcal{G} . If we start with $\sigma_\rho = 1$ in T_3 , the number subtrees of the root with leaf configurations in \mathcal{G} is distributed binomially and will be about kp . By Chernoff bounds, and the bound on p from (8),

$$\mathbb{P}(\text{Bin}(k, p) < e^\beta \ln \ln k - 2\sqrt{e^\beta \ln \ln k}) < \frac{1}{3}.$$

Finally, by the definition of \mathcal{G} ,

$$\mathbb{P}_{T_2}^0[\sigma_{u_i} = 0 | \sigma(L) \in \mathcal{G}] = \frac{1}{2} \left(1 + \frac{1}{1+2\lambda} \right)$$

and hence,

$$\begin{aligned} \mathbb{E}_{T_3}^1[\mathbb{P}[\sigma_\rho = 1 | \sigma(L)]] &= \mathbb{E}_{T_3}^1[1 - \mathbb{P}[\sigma_\rho = 0 | \sigma(L)]] \\ &\leq \left(1 - \frac{1}{1 + \lambda[2(1 - o_k(1))]^{-(e^\beta \ln \ln k - 2\sqrt{e^\beta \ln \ln k})}} \right) \frac{2}{3} + \frac{1}{3}. \end{aligned}$$

By taking k large enough above, we conclude that for β and large enough k ,

$$\mathbb{E}_{T_3}^1[\mathbb{P}[\sigma_\rho = 1 | \sigma(L)]] \leq \frac{1}{2}. \quad \square$$

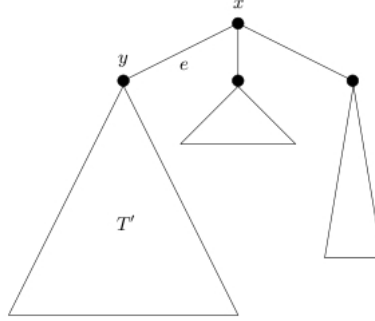
Lemma 2.4. *Let $\beta > \ln 2 - \ln \ln 2$ and $\omega = \frac{1}{k} [\ln k + \ln \ln k - \ln \ln \ln k - \beta]$. For k large enough,*

$$\overline{X}(3) \leq \frac{\omega}{2}.$$

Proof. By part c) of Lemma 2.2, and part c) of Proposition 2.3,

$$\begin{aligned} \overline{X}(3) &= \frac{1}{\pi_{01}^2} \left(\frac{\mathbb{E}_{T_3}^1[\mathbb{P}[\sigma_\rho = 1 | \sigma(L)]]}{\pi_1} - 1 \right) \\ &\leq \frac{1}{\pi_{01}^2} \left(\frac{1}{2\pi_1} - 1 \right) \\ &\leq \frac{\omega}{2}. \quad \square \end{aligned}$$

Next, we present a recursion for \overline{X} and complete the proof of the main result. The development of the recursion follows the steps in [5] closely so we follow their notation and omit some of the calculations in this short version.

FIGURE 1. A finite tree T

Magnetisation of a child. With T and x as defined previously, let y be a child of x and let T' be the subtree of T rooted at y (see Figure 1). Let A' be the restriction of A to the leaves of T' . Let $Y = Y(A')$ denote the magnetization of y .

Lemma 2.5. *We have*

- a) $\mathbb{E}_T^1[Y] = \theta \mathbb{E}_{T'}^1[Y]$ and $\mathbb{E}_T^0[Y] = \theta \mathbb{E}_{T'}^0[Y]$.
- b) $\mathbb{E}_T^1[Y^2] = (1 - \theta) \mathbb{E}_{T'}[Y^2] + \theta \mathbb{E}_{T'}^1[Y^2]$.
- c) $\mathbb{E}_T^0[Y^2] = (1 - \theta) \mathbb{E}_{T'}[Y^2] + \theta \mathbb{E}_{T'}^0[Y^2]$.

The proof follows from the first part of Lemma 2.2 and the Markov property when we condition on x .

Next, we can write the effect on the magnetization of adding an edge to the root and merging roots of two trees as follows. Referring to Figure 2, let T' (resp. T'') be a finite tree rooted at y (resp. z) with the channel on all edges being given M , leaf states A (resp. A'') and weighted magnetisation at the root Y (resp. Z). Now add an edge (\hat{y}, z) to T'' to obtain a new tree \hat{T} . Then merge \hat{T} with T' by identifying $y = \hat{y}$ to obtain a new tree T . To avoid ambiguities, denote by x the root of T and X the magnetization of the root of T . We let $A = (A', A'')$ be the leaf state of T . Let \hat{Y} be the magnetization of the root of \hat{T} .

Note: In the above construction, the vertex y is a vertex “at the same level” as x , and not a child of x as it was in Lemma 2.5.

Lemma 2.6. *With the notation above, $\hat{Y} = \theta Z$.*

The proof follows by applying Bayes rule, the Markov property and Lemma 2.2. These facts also imply that

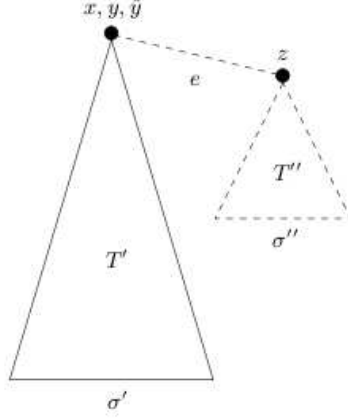


FIGURE 2. The tree T after obtained after merging T' and T'' . The dashed subtree is \hat{T} .

Lemma 2.7. *For any tree \hat{T} ,*

$$X = \frac{Y + \hat{Y} + \Delta Y \hat{Y}}{1 + \pi_{01} Y \hat{Y}}.$$

With these lemmas in hand we can use the following relation to derive a recursive upper bound on the second moments. We will use the expansion

$$\frac{1}{1+r} = 1 - r + r^2 \frac{1}{1+r}.$$

Taking $r = \pi_{01} Y \hat{Y}$, by Lemma 2.7 we have

$$\begin{aligned} X &= (Y + \hat{Y} + \Delta Y \hat{Y}) \left[1 - \pi_{01} Y \hat{Y} + (\pi_{01} Y \hat{Y})^2 \frac{1}{1 + \pi_{01} Y \hat{Y}} \right] \\ &= Y + \hat{Y} + \Delta Y \hat{Y} - \pi_{01} Y \hat{Y} (Y + \hat{Y} + \Delta Y \hat{Y}) + (\pi_{01})^2 (Y \hat{Y})^2 X \\ (9) \quad &\leq Y + \hat{Y} + \Delta Y \hat{Y} - \pi_{01} Y \hat{Y} (Y + \hat{Y} + \Delta Y \hat{Y}) + (\pi_{01})^2 (Y \hat{Y})^2 \end{aligned}$$

where the last inequality follows since $X \leq 1$ with probability 1.

Let $\rho' = \overline{Y}_1 / \overline{Y}$ and $\rho'' = \overline{Z}_1 / \overline{Z}$. Below, the moments \overline{Y} etc. are defined according to the appropriate measures over the tree rooted at y (i.e. T') etc.

By applying Lemmas 2.2, 2.5 and 2.6, we have the following relations.

$$\begin{aligned} \mathbb{E}_T^1[X] &= \pi_{01} \overline{X}, \quad \mathbb{E}_T^1[Y] = \pi_{01} \overline{y}, \quad \mathbb{E}_T^1[Y^2] = \overline{Y} \rho' \\ (10) \quad \mathbb{E}_T^1[\hat{Y}] &= \pi_{01} \theta^2 \overline{Z}, \quad \mathbb{E}_T^1[\hat{Y}^2] = \theta^2 \overline{Z} ((1 - \theta) + \theta \rho''). \end{aligned}$$

Applying $(\pi_{01})^{-1}E_T^1[\cdot]$ to both sides of (9), we obtain the following.

$$\begin{aligned}\overline{X} &\leq \overline{Y} + \theta^2 \overline{Z} + \Delta \pi_{01} \overline{Y} \overline{Z} - \pi_{01} \theta^2 \overline{Y} \overline{Z} \rho' - \pi_{01} \theta^2 \overline{Y} \overline{Z} ((1 - \theta) + \theta \rho'') \\ &\quad - \Delta \theta^2 \overline{Y} \overline{Z} \rho' ((1 - \theta) + \theta \rho'') + \pi_{01} \theta^2 \overline{Y} \overline{Z} \rho' ((1 - \theta) + \theta \rho'') \\ &= \overline{Y} + \theta^2 \overline{Z} - \pi_{01} \theta^2 \overline{Y} \overline{Z} [\mathcal{A} - \Delta \mathcal{B}]\end{aligned}$$

where

$$\mathcal{A} = \rho' + (1 - \rho')[(1 - \theta) + \theta \rho''],$$

$$\text{and } \mathcal{B} = 1 - (\pi_{01})^{-1} \rho' [(1 - \theta) + \theta \rho''] = 1 - \frac{\omega}{1 + \omega} \rho' [(1 - \theta) + \theta \rho''] .$$

If $\mathcal{A} - \Delta \mathcal{B} \geq 0$, this would already give a sufficiently good recursion to show that $\overline{X}(n)$ goes to 0, so we will assume is negative and try to get a good (negative) lower bound. First note that by their definition $\rho', \rho'' \geq 0$. Further since $\overline{Y} = \pi_1 \overline{Y}_1 + \pi_0 \overline{Y}_0$,

$$\rho' \leq (\pi_1)^{-1} = \frac{1 + 2\omega}{\omega} .$$

Similarly,

$$\rho'' \leq (\pi_1)^{-1} = \frac{1 + 2\omega}{\omega} .$$

Since $\mathbb{E}_T^1[\hat{Y}^2]$ and $\overline{Z} \geq 0$, it follows from (10) that $(1 - \theta) + \theta \rho'' \geq 0$. Together with the fact that $\rho' \geq 0$, this implies that $\mathcal{B} \leq 1$.

Since \mathcal{A} is multi-linear in (ρ', ρ'') , to minimize it, its sufficient to consider the extreme cases. When $\rho' = 0$, \mathcal{A} is minimized at the upper bound of ρ'' and hence

$$\mathcal{A} \geq 1 - \pi_{01} \frac{\omega}{1 + \omega} = 0.$$

When $\rho' = (\pi_1)^{-1}$,

$$\mathcal{A} = (\pi_1)^{-1} + (1 - (\pi_1)^{-1})[1 - \theta(1 - \rho'')] \geq 0.$$

Hence, we have

$$\overline{X} \leq \overline{Y} + \theta^2 \overline{Z} + \frac{1}{1 + \omega} \overline{Y} \overline{Z}.$$

Applying this recursively to the tree, we obtain the following recursion for the moments.

$$\overline{X} \leq (1 + \omega) \theta^2 \left[\left(1 + \frac{\overline{Z}}{1 + \omega} \right)^k - 1 \right] .$$

We bound the $(1 + x)^k - 1$ term as,

$$|(1 + x)^k - 1| \leq e^{|x|k} - 1 = \int_0^{|x|k} e^s ds \leq e^{|x|k} k |x|$$

and this implies the following recursion.

Theorem 2.8. *If for some n , $\overline{X}(n) \leq \frac{\omega}{2}$, we have that*

$$\overline{X}(n+1) \leq \omega^2 e^{\frac{1}{2}\omega k} k \overline{X}(n).$$

Thus if $\omega^2 e^{\frac{1}{2}\omega k} k < 1$ then it follows from the recursion that

$$(11) \quad \lim_n \overline{X}(n) = 0.$$

When $\omega = \frac{1}{k} [\ln k + \ln \ln k - \ln \ln \ln k - \beta]$ and $\beta > \ln 2 - \ln \ln 2$, by Lemma 2.4, for k large enough, $\overline{X}(3) \leq \frac{\omega}{2}$. Hence by equation (11) we have that $\overline{X}(n) \rightarrow 0$ and so by Proposition 2.1 we have non-reconstruction. Since reconstruction is monotone in λ and hence in ω it follows that we have non-reconstruction for $\omega \leq \bar{\omega}$ for large k . This completes the proof of Theorem 1.1.

REFERENCES

- [1] D. Achlioptas and A. Coja-Oghlan. Algorithmic barriers from phase transitions. *Proceedings of IEEE FOCS 2008*, 793–802, 2008.
- [2] N. Berger, C. Kenyon, E. Mossel, and Y. Peres, Glauber dynamics on trees and hyperbolic graphs. *Probability Theory and Related Fields* **131** (2005), 311–340.
- [3] N. Bhatnagar, J. Vera, E. Vigoda and D. Weitz. Reconstruction for colorings on trees. To appear in *SIAM Journal on Discrete Mathematics*.
- [4] P. M. Bleher, J. Ruiz, and Zagrebnov V. A. On the purity of limiting gibbs state for the Ising model on the Bethe lattice. *Journal of Statistical Physics*, **79** (1995), 473–482.
- [5] C. Borgs, J. Chayes, E. Mossel and S. Roch. The Kesten-Stigum reconstruction bound is tight for roughly symmetric binary channels. *Proceedings of IEEE FOCS 2006*, 518–530.
- [6] C. Cooper, A. Frieze, B. Reed and O. Riordan. Random regular graphs of non-Constant degree: Independence and Chromatic Number. *Combinatorics, Probability and Computing*, **11** (2002), 323–341.
- [7] G. Brightwell and P. Winkler. A second threshold for the hard-core model on a Bethe lattice. *Random structures and algorithms*, **24** (2004), 303–314.
- [8] C. Daskalakis, E. Mossel, and S. Roch. Optimal phylogenetic reconstruction. *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, 159–168, 2006.
- [9] J. Ding E. Lubetzky and Y. Peres. Mixing time of critical Ising model on trees is polynomial in the height. *Communications in Mathematical Physics*, **295** (2010), 161–207.
- [10] W. Evans, C. Kenyon, Y. Peres, and L. J. Schulman. Broadcasting on trees and the Ising model. *Annals of Applied Probability*, **10** (2000), 410–433.
- [11] J. Felsenstein. Inferring Phylogenies. *Sinauer*, New York, New York, 2004.
- [12] H. O. Georgii. *Gibbs measures and phase transitions*. Walter de Gruyter, Berlin, 1988.
- [13] S. Janson and E. Mossel. Robust reconstruction on trees is determined by the second eigenvalue. *Annals of Probability*, **32** (2004), 2630–2649.
- [14] F. Krzakala, A. Montanari, F. Ricci-Tersenghi, G. Semerjian, and L. Zdeborová, Gibbs states and the set of solutions of random constraint satisfaction problems. *Proceedings of the National Academy of Sciences* **104** (2007), 10318.
- [15] H. Kesten and B. P. Stigum, Additional limit theorems for indecomposable multidimensional Galton-Watson processes, *Annals of Mathematical Statistics* **37** (1966), 1463–1481.
- [16] J. Martin. Reconstruction thresholds on regular trees. in Discrete Random Walks, DRW03, C. Banderier and C. Krattenthaler eds., *Discrete Mathematics and Theoretical Computer Science Proceedings*, 191–204, 2003.

- [17] F. Martinelli, A. Sinclair, and D. Weitz. Fast mixing for independent sets, colorings, and other models on trees. *Proceedings of ACM-SIAM SODA*, 449–458, 2004.
- [18] F. Martinelli, A. Sinclair, and D. Weitz. Glauber dynamics on trees: boundary conditions and mixing time. *Communications in Mathematical Physics*, **250** (2004), 301–334.
- [19] M. Mézard and A. Montanari. Reconstruction on trees and spin glass transition. *Journal of Statistical Physics*, **124** (2006), 1317–1350.
- [20] A. Montanari, R. Restrepo, and P. Tetali. *Reconstruction and Clustering Thresholds for Random Constraint Satisfaction Problems*. Preprint 2009.
- [21] E. Mossel. Survey: Information flow on trees. In J. Neštril and P. Winkler, editors, *Graphs, Morphisms and Statistical Physics. DIMACS series in discrete mathematics and theoretical computer science*, pages 155–170. Amer. Math. Soc., 2004.
- [22] E. Mossel. Reconstruction on trees: beating the second eigenvalue. *Annals of Applied Probability*, **11** (2001), 285–300.
- [23] E. Mossel and Y. Peres. Information flow on trees. *Annals of Applied Probability*, **13** (2003), 817–844.
- [24] A. Sly. Reconstruction of random colourings. *Communications of Mathematical Physics*, **288** (2009), 943–961.
- [25] A. Sly. Reconstruction of symmetric Potts Models. *Proceedings of the 41st ACM Symposium on Theory of Computing* 581–590, 2009.
- [26] P. Tetali, J. Vera, E. Vigoda and L. Yang. Phase Transition for the Mixing Time of the Glauber Dynamics for Coloring Regular Trees. *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms*, 1646–1656, 2010.
- [27] N. Wormald. Differential equations for random processes and random graphs. *Annals of Applied Probability*, **5** (1995), 1217–1235.

NAYANTARA BHATNAGAR
 SCHOOL OF COMPUTER SCIENCE AND ENGINEERING
 HEBREW UNIVERSITY OF JERUSALEM
 JERUSALEM, 91904, ISRAEL.

E-mail address: `nayantara@cs.huji.ac.il`

ALLAN SLY
 THEORY GROUP, MICROSOFT RESEARCH
 ONE MICROSOFT WAY
 REDMOND, WA, 98052, USA.

E-mail address: `allansly@microsoft.com`

PRASAD TETALI
 SCHOOL OF MATHEMATICS AND SCHOOL OF COMPUTER SCIENCE
 GEORGIA INSTITUTE OF TECHNOLOGY
 ATLANTA, GA, 30332, USA.

E-mail address: `tetali@math.gatech.edu`